Lecture 19. Polynomials and linear transformations

Ex Parametrize all polynomials $p(t) \in \mathbb{P}_3$ with

$$p(-2) = 2, \quad p(0) = 4, \quad p(1) = 8.$$

Sol Consider the linear transformation $T : \mathbb{P}_3 \longrightarrow \mathbb{R}^3$ given by

$$T(p(t)) = \begin{bmatrix} p(-2) \\ p(0) \\ p(1) \end{bmatrix}.$$

We solve the equation $T(p(t)) = \vec{b}$ with

$$\vec{b} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}.$$

The standard matrix has columns $T(1), T(t), T(t^2), T(t^3)$.

$$p(t) = 1 : \; p(-2) = 1, \; p(0) = 1, \; p(1) = 1 \implies T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$p(t) = t : \; p(-2) = -2, \; p(0) = 0, \; p(1) = 1 \implies T(t) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

$$p(t) = t^2 : \; p(-2) = 4, \; p(0) = 0, \; p(1) = 1 \implies T(t^2) = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}.$$

$$p(t) = t^3 : \; p(-2) = -8, \; p(0) = 0, \; p(1) = 1 \implies T(t^3) = \begin{bmatrix} -8 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the standard matrix is

$$A = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Now we can convert the equation $T(p(t)) = \vec{b}$ into a matrix equation $A\vec{x} = \vec{b}$ by setting $\vec{x} = [p(t)]$.

$$\underset{A \qquad\quad \vec{b}}{\left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 2 \\ 1 & 0 & 0 & 0 & 4 \\ 1 & 1 & 1 & 1 & 8 \end{array}\right]} \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\Rightarrow \left\{\begin{array}{l} x_1 \qquad\quad = 4 \\ x_2 + 2x_4 = 3 \\ x_3 - \ \ x_4 = 1 \end{array}\right. \Rightarrow \left\{\begin{array}{l} x_1 = 4 \\ x_2 = 3 - 2x_4 \\ x_3 = 1 + \ \ x_4 \end{array}\right. \underset{x_4 = c}{\Rightarrow} \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3-2c \\ 1+c \\ c \end{bmatrix}$$

As we set $\vec{x} = [p(t)]$, we find

$$p(t) = \boxed{4 + (3-2c)t + (1+c)t^2 + ct^3}$$

Note (1) This example demonstrates polynomial interpolation, which refers to the process of finding polynomials whose graphs pass through specific points.

(2) We may write the solution as

$$\begin{aligned} p(t) &= (4 + 3t + t^2) + c(-2t + t^2 + t^3) \\ &= (4 + 3t + t^2) + ct(t-1)(t+2). \end{aligned}$$

The first polynomial $4 + 3t + t^2$ is given by the last column in the RREF, whereas the second polynomial $t(t-1)(t+2)$ has roots at $t = \underbrace{-2, 0, 1}_{\text{specified inputs}}$.

Ex Find the polynomial solution of the differential equation

$$p(t)+2p'(t)+3p''(t)=2t^2+3t+5.$$

Sol Consider the linear transformation $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$ given by

$$T(p(t)) = p(t)+2p'(t)+3p''(t).$$

We solve the equation $T(p(t)) = 2t^2+3t+5$.

The standard matrix has columns $[T(1)], [T(t)], [T(t^2)]$.

* $T(1)$, $T(t)$, $T(t^2)$ themselves are not column vectors

$p(t)=1$: $p'(t)=0$, $p''(t)=0 \Longrightarrow T(1)=1+2\cdot 0+3\cdot 0=1$

$p(t)=t$: $p'(t)=1$, $p''(t)=0 \Longrightarrow T(t)=t+2\cdot 1+3\cdot 0=t+2$

$p(t)=t^2$: $p'(t)=2t$, $p''(t)=2 \Longrightarrow T(t^2)=t^2+2\cdot 2t+3\cdot 2=t^2+4t+6$

Hence the standard matrix is

$$A=\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

* Since T takes values in $\mathbb{P}_2$, each coordinate vector has 3 entries (with standard basis of $\mathbb{P}_2$ given by $1, t, t^2$)

Now we can convert the equation $T(p(t)) = 2t^2+3t+5$ into a matrix equation $A\vec{x} = [2t^2+3t+5]$ by setting $\vec{x}=[p(t)]$.

$$\underset{A}{\left[\begin{array}{ccc|c} 1 & 2 & 6 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]} \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\Longrightarrow x_1=3$, $x_2=-5$, $x_3=2$

As we set $\vec{x}=[p(t)]$, we find $p(t)=\boxed{3+5t+2t^2}$